

Generating Univariate Fractional Integration within a Large VAR(1), Supplementary Appendix

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1 Examples relating to Remark 2

1.1 Example of ARFIMA($p, \delta, 0$)

Consider \mathbf{y}_t such that $\mathbf{F}_n(L) \mathbf{y}_t = \mathbf{x}_t$, with

$$\mathbf{F}_n(L) = \mathbf{I}_n - \begin{bmatrix} a & 0 & \mathbf{0}'_{n-2} \\ b & c & \mathbf{e}' \\ \mathbf{d} & \mathbf{0}_{n-2} & \mathbf{0}_{(n-2) \times (n-2)} \end{bmatrix} L,$$

where a, b, c are scalar, \mathbf{e} and \mathbf{d} denote vectors of dimension $(n-2)$, and $\mathbf{0}_{n-2}$ is a vector of zeros of dimension $(n-2)$. The FER is

$$(1-aL)(1-cL) \mathbf{y}_t = \begin{bmatrix} 1-cL & 0 & \mathbf{0}_{1 \times (n-2)} \\ (b-\mathbf{e}'\mathbf{d}L)L & 1-cL & (1-aL)L\mathbf{e}' \\ (1-cL)L\mathbf{d} & \mathbf{0}_{(n-2) \times 1} & (1-aL)(1-cL)\mathbf{I}_{n-2} \end{bmatrix} \mathbf{x}_t$$

so for $j=1$, $(1-aL)(1-cL)y_{1t} = (1-cL)x_{1t}$, i.e. cancelling the common factor $(1-aL)y_{1t} = x_{1t}$. As $n \rightarrow \infty$, under the assumptions of Theorem 1 with $j=1$,

$$(1-aL)y_{1t} \xrightarrow{L} \Delta^{-\delta} \epsilon_{1t}.$$

This example generalizes to

$$\mathbf{F}_n(L) = \mathbf{I}_n - \begin{bmatrix} a & \mathbf{0}_{1 \times (n-1)} \\ \mathbf{b}_{(n-1) \times 1} & \mathbf{C}_{(n-1) \times (n-1)} \end{bmatrix} L$$

which yields an asymptotic ARFIMA($1, \delta, 0$) for y_{1t} .

1.2 Example of ARFIMA(p, δ, q)

Let $\mathbf{F}_n(L) \mathbf{y}_t = \mathbf{x}_t$ and assume

$$\mathbf{F}_n(L) = \begin{bmatrix} h(L) & 1 & \mathbf{0}'_{n-2} \\ -\alpha(L) + h(L)\beta(L) & \beta(L) & \mathbf{0}'_{n-2} \\ \mathbf{0}_{n-2} & \mathbf{0}_{n-2} & \mathbf{I}_{n-2} \end{bmatrix},$$

where $\mathbf{0}_{n-2}$ denotes a vector of zeros of dimension $(n-2)$, $\alpha(L)$ and $\beta(L)$ are invertible scalar polynomials of fixed orders p and q with no common roots, $h(L)$ denotes a scalar polynomial. Using the FER for \mathbf{y}_t , we obtain

$$\alpha(L) \mathbf{y}_t = \begin{bmatrix} \beta(L) & 1 & \mathbf{0}'_{n-2} \\ -\alpha(L) + h(L)\beta(L) & h(L) & \mathbf{0}'_{n-2} \\ \mathbf{0}_{n-2} & \mathbf{0}_{n-2} & \alpha(L) \end{bmatrix} \mathbf{x}_t.$$

Under the conditions of Theorem 1, with $j=1$, and as $n \rightarrow \infty$, $x_{1t} \xrightarrow{L} \Delta^{-\delta} \epsilon_{1t}$ so $z_t = y_{1t} - \alpha(L)^{-1} x_{2t}$ tends to a process solution to

$$\alpha(L) z_t = \beta(L) \Delta^{-\delta} \epsilon_{1t},$$

i.e. an ARFIMA(p, δ, q).

2 A useful Lemma

In subsequent proofs, we use the following lemma.

Lemma L Under the assumptions of both Examples 1 and 2 and as $n \rightarrow \infty$, the coefficients of \mathbf{T}_n^* satisfy for all k , $-n < k < n$, $t_k^{*(n-1)} - t_k^{*(n)} \sim \frac{1}{n} t_k^{*(n)}$ for all n such that $(n-1)/4 \in \mathbb{N}$.

Proof. Recall that $t_k^{*(n)} = \operatorname{Re} \left[\frac{1}{n} \sum_{\ell=0}^{n-1} g \left(\delta_n, e^{i \frac{2\pi\ell}{n}} \right) e^{-2i\pi\ell k/n} \right]$, where $\operatorname{Re}[\cdot]$ denotes the real part, so

$$\begin{aligned} & t_k^{*(n-1)} - t_k^{*(n)} \\ &= \frac{1}{n(n-1)} \operatorname{Re} \left[n \sum_{\ell=0}^{n-2} g \left(\delta_{n-1}, e^{i \frac{2\pi\ell}{n-1}} \right) e^{-2i\pi\ell k/(n-1)} - (n-1) \sum_{\ell=0}^{n-1} g \left(\delta_n, e^{i \frac{2\pi\ell}{n}} \right) e^{-2i\pi\ell k/n} \right] \\ &= \frac{1}{n-1} \operatorname{Re} \left[\sum_{\ell=0}^{n-2} \left[g \left(\delta_{n-1}, e^{i \frac{2\pi\ell}{n-1}} \right) e^{-2i\pi\ell k/(n-1)} - g \left(\delta_n, e^{i \frac{2\pi\ell}{n}} \right) e^{-2i\pi\ell k/n} \right] \right] \\ &+ \frac{1}{n(n-1)} \sum_{\ell=0}^{n-1} g \left(\delta_n, e^{i \frac{2\pi\ell}{n}} \right) e^{-2i\pi\ell k/n} - \frac{1}{n} g \left(\delta_n, e^{i \frac{2\pi(n-1)}{n}} \right) e^{-2i\pi(n-1)k/n}. \end{aligned}$$

Hence, as $n^{-1} = (n-1)^{-1} (1 - n^{-1})$,

$$\begin{aligned} & t_k^{*(n-1)} - t_k^{*(n)} \\ &= \frac{1}{n-1} \operatorname{Re} \left[\sum_{\ell=0}^{n-2} \left[g \left(\delta_{n-1}, e^{i \frac{2\pi\ell}{n-1}} \right) - g \left(\delta_n, e^{i \frac{2\pi\ell}{n}} \right) e^{2i\pi\ell k \frac{1}{n(n-1)}} \right] e^{-2i\pi\ell k/(n-1)} \right] \\ &+ \frac{1}{n-1} t_k^{*(n)} - \frac{1}{n} \operatorname{Re} \left[g \left(\delta_n, e^{i \frac{2\pi(n-1)}{n}} \right) e^{-2i\pi(n-1)k/n} \right]. \end{aligned}$$

Recall the definition $g(\delta, e^{i\omega}) = 1_{\{0 \leq u < \pi\delta\}} + 1_{\{\pi(\frac{3}{2}-\delta) < u \leq \frac{3\pi}{2}\}}$ for $\omega = u \bmod 2\pi \geq 0$, so g is real, and

$$g \left(\delta_n, e^{i \frac{2\pi\ell}{n}} \right) = 1_{\{0 \leq \frac{2\pi\ell}{n} < \pi\delta_n\}} + 1_{\{\pi(\frac{3}{2}-\delta_n) < \frac{2\pi\ell}{n} \leq \frac{3\pi}{2}\}}.$$

We assumed $\delta_n = \delta + o(n^{-2})$ with (δ_n) a nondecreasing sequence. Hence, for all δ and n large enough, for all $\ell < n$

$$\frac{2\ell}{n-1} < \delta_{n-1} \Leftrightarrow \frac{2\ell}{n} < \delta_n \Leftrightarrow \frac{2\ell}{n} < \delta,$$

i.e. $1_{\{0 \leq \frac{2\pi\ell}{n-1} < \pi\delta_{n-1}\}} = 1_{\{0 \leq \frac{2\pi\ell}{n} < \pi\delta_n\}} = 1_{\{0 \leq \frac{2\pi\ell}{n} < \pi\delta\}}$. Also, for n large enough,

$$\left(\frac{3}{2} - \delta_{n-1} \right) < \frac{2\ell}{n-1} \Leftrightarrow \left(\frac{3}{2} - \delta_n \right) < \frac{2\ell}{n} \Leftrightarrow \left(\frac{3}{2} - \delta \right) < \frac{2\ell}{n}$$

so $1_{\{\pi(\frac{3}{2}-\delta_{n-1}) < \frac{2\pi\ell}{n-1}\}} = 1_{\{\pi(\frac{3}{2}-\delta_n) < \frac{2\pi\ell}{n}\}} = 1_{\{\pi(\frac{3}{2}-\delta) < \frac{2\pi\ell}{n}\}}$. Finally, for all n , let $(m, s) \in \mathbb{N} \times \{0, 1, 2, 3\}$ such that $n = 4m + s$. Clearly $\frac{2\pi\ell}{n-1} \leq \frac{3\pi}{2} \Rightarrow \frac{2\pi\ell}{n} \leq \frac{3\pi}{2}$, now, for the converse,

$$\frac{2\pi\ell}{n} < \frac{3\pi}{2} \Leftrightarrow \ell < \frac{3n}{4} = 3m + \frac{3}{4}s,$$

hence, since ℓ is an integer $\ell \leq \frac{3n}{4} \Rightarrow \ell \leq 3m + s_s^*$, where $s_0^* = s_1^* = 0$ and $s_s^* = s - 1$ for $s \in \{2, 3\}$.

Therefore

$$\frac{\ell}{n-1} \leq \frac{3m + s_s^*}{4m + s - 1} = \frac{3}{4} + \frac{(4s_s^* - 3s + 3)/4}{4m + s - 1},$$

where $4s_s^* - 3s + 3 \leq 0$ for $s = 1$. Hence, for $(n-1)/4 \in \mathbb{N}$,

$$1_{\left\{\frac{2\pi\ell}{n-1} \leq \frac{3\pi}{2}\right\}} = 1_{\left\{\frac{2\pi\ell}{n} \leq \frac{3\pi}{2}\right\}}.$$

Therefore, there exists N such that if $n > N$ and $(n-1)/4 \in \mathbb{N}$, $g\left(\delta_{n-1}, e^{i\frac{2\pi\ell}{n-1}}\right) = g\left(\delta_n, e^{i\frac{2\pi\ell}{n}}\right) = g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right)$ so that

$$\begin{aligned} t_k^{*(n-1)} - t_k^{*(n)} &= \frac{1}{n-1} \operatorname{Re} \left[\sum_{\ell=0}^{n-2} g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right) \left[e^{-2i\pi\ell k/n} - e^{-2i\pi\ell k/(n-1)} \right] \right] \\ &\quad + \frac{1}{n-1} t_k^{*(n)} - \frac{1}{n} g\left(\delta_n, e^{i\frac{2\pi(n-1)}{n}}\right) \cos \frac{2\pi(n-1)k}{n}. \end{aligned}$$

The definition of $g(\cdot, \cdot)$ implies that as $n \rightarrow \infty$, $g\left(\delta_n, e^{i\frac{2\pi(n-1)}{n}}\right) \rightarrow \lim_{u \rightarrow 2\pi} g(\delta, e^{iu})$ which is zero, and in particular as $g(d, u)$ is identically zero in a neighborhood $u \in [2\pi - \epsilon, 2\pi)$ for d in a neighborhood of δ ($\delta > 0$). Hence, there exists $M > 0$ such that for $n > M$, $g\left(\delta_n, e^{i\frac{2\pi(n-1)}{n}}\right) = 0$. Now, if $n > \max(M, N)$ and $(n-1)/4 \in \mathbb{N}$,

$$t_k^{*(n-1)} - t_k^{*(n)} = \frac{1}{n-1} \operatorname{Re} \left[\sum_{\ell=0}^{n-2} g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right) \left[e^{-2i\pi\ell k/n} - e^{-2i\pi\ell k/(n-1)} \right] \right] + \frac{1}{n-1} t_{-k}^{*(n)}$$

with $(n-1)^{-1} \operatorname{Re} \left[\sum_{\ell=0}^{n-2} g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right) \left[e^{-2i\pi\ell k/n} - e^{-2i\pi\ell k/(n-1)} \right] \right] \sim t_k^{*(n)} + t_k^{*(n)}/(n-1) - t_k^{*(n-1)}$. Hence

$$t_k^{*(n-1)} - t_k^{*(n)} \sim \frac{1}{n} t_k^{*(n)} \quad (\text{S-1})$$

for n such that $(n-1)/4 \in \mathbb{N}$. \square

3 Proofs relative to Example 1

We collect here the proofs related to Example 1 that shows that Assumptions T and P are satisfied for $\mathbf{A}_n = \mathbf{T}_n^* + \eta_n \mathbf{D}_n$, where \mathbf{T}_n^* , η_n and \mathbf{D}_n are specified as in Section ??.

3.1 Proof of the validity of Assumption T for the matrix \mathbf{T}_n

Assumptions T(i), and T(iii.a) follow from the definitions of g and δ_n . To prove that Assumption T(ii) holds, we need to show that $\mathbf{T}_{d,n}$ belongs to the Wiener class for all $d \in (0, 1)$. This follows from the fact that the derivative $\frac{\partial}{\partial \omega} g(d, e^{i\omega})$ is continuous at $\omega = 0$ for all $d > 0$. Hence, the Fourier series of $g(d, e^{i\omega})$ is absolutely summable at $\omega = 0$ (see Whittaker, 1930-31), i.e., $\lim_{n \rightarrow \infty} \sum_{k=-(n-1)}^{n-1} |t_{d,k}| < \infty$. Hence Assumption T(ii) holds. Now, for Assumption T(iii.b), notice that for $\delta_n \leq \delta$,

$$t_{\delta,k}^{(n)} - t_k^{(n)} = \frac{1}{n} \sum_{\ell=\lceil n\delta_n/2 \rceil}^{\lceil n\delta/2 \rceil - 1} e^{-2i\pi k\ell/n} + \frac{1}{n} \sum_{\ell=\lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor}^{\lfloor \frac{3n}{4} - \frac{n\delta}{2} \rfloor} e^{-2i\pi k\ell/n}, \quad (\text{S-2})$$

where the first sum becomes identically zero when $\lceil n\delta_n/2 \rceil - 1 = \lceil n\delta/2 \rceil - 1$, i.e. $\lceil n\delta/2 \rceil - 1 < n\delta_n/2$. It is the case when $\frac{\lceil n\delta/2 \rceil - 1}{n/2} < \delta - (\delta - \delta_n)$, i.e. $\delta - \delta_n < \frac{\lceil n\delta/2 \rceil - n\delta/2 - 1}{n/2} = O(n^{-1})$. Since $\delta - \delta_n = o(n^{-2})$, there exists $N > 0$ such that the latter expression holds for $n > N$. The second term in Expression (S-2) is itself identically zero when $\lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor = \lfloor \frac{3n}{4} - \frac{n\delta}{2} \rfloor$, i.e. for $\delta = 1/2$, $\lfloor \frac{n}{2} + \frac{n(\delta - \delta_n)}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$ which holds for $\frac{n(\delta - \delta_n)}{2} < 1/2$ since $\frac{n}{2} - \lfloor \frac{n}{2} \rfloor \in \{0, \frac{1}{2}\}$. It suffices that $n(\delta - \delta_n) < 1$, which holds for n large enough. Therefore $n^2 \left(t_{\delta,k}^{(n)} - t_k^{(n)} \right)$ also becomes identically zero for n large enough, i.e. T(iii.b) holds.

3.2 Proof of the validity of Assumptions P(i)-(iv) for the matrix $\mathbf{A}_n = \mathbf{T}_n^* + \eta_n \mathbf{D}_n$

We let $j = 1$ without loss of generality and start by showing that Assumption P(ii) holds. The elements of Σ_n are constant hence we set it to \mathbf{I}_n without loss of generality.

Elements $\widetilde{\mathbf{B}}_n(z)_{1k}$, for $k = 1, \dots, n$, of the first row of $\widetilde{\mathbf{B}}_n(z)$, satisfy $\widetilde{\mathbf{B}}_n(z)_{1k} = (-1)^{k+1} \det(\mathbf{CoB}_n(z)_{1k})$, where $\mathbf{CoB}_n(z)_{\ell k}$ is the (ℓ, k) entry of the matrix of cofactors of $\mathbf{B}_n(z)$. We consider first $\mathbf{CoB}_n(z)_{11}$ which is

$$\mathbf{CoB}_n(z)_{11} = \begin{bmatrix} 1 - t_0^{*(n)} z & - (t_1^{*(n)} + \eta_n \gamma_{23}^{(n)}) z & \cdots & - (t_{n-2}^{*(n)} + \eta_n \gamma_{2n}^{(n)}) z \\ - (t_1^{*(n)} + \eta_n \gamma_{32}^{(n)}) z & 1 - t_0^{*(n)} z & \ddots & \vdots \\ \vdots & \ddots & \ddots & - (t_1^{*(n)} + \eta_n \gamma_{(n-1)n}^{(n)}) z \\ - (t_{n-2}^{*(n)} + \eta_n \gamma_{n2}^{(n)}) z & \cdots & - (t_1^{*(n)} + \eta_n \gamma_{n(n-1)}^{(n)}) z & 1 - t_0^{*(n)} z \end{bmatrix},$$

where $\gamma_{\ell k}$ denotes the (ℓ, k) entry of \mathbf{D}_n . Denoting respectively by $\mathbf{T}_n^{*(1)}$ and $\mathbf{D}_n^{(1)}$ the submatrices of \mathbf{T}_n^* and \mathbf{D}_n of dimension $n - 1$ obtained by removing their first row and first column, $\mathbf{CoB}_n(z)_{11}$ can be written in a matrix form as

$$\mathbf{CoB}_n(z)_{11} = \mathbf{B}_{n-1}(z) - \left(\mathbf{T}_{n-1}^* - \mathbf{T}_n^{*(1)} + \eta_{n-1} \mathbf{D}_{n-1} - \eta_n \mathbf{D}_n^{(1)} \right) z. \quad (\text{S-3})$$

Now Lemma L implies that for $(n-1)/4 \in \mathbb{N}$, $\mathbf{CoB}_n(z)_{11} = \mathbf{I}_{n-1} - z \mathbf{T}_{n-1}^* + \mathbf{O}\left(\frac{z}{n} \mathbf{T}_{n-1}^*\right)$, hence $\det \mathbf{CoB}_n(z)_{11} \sim \det(\mathbf{I}_{n-1} - z \mathbf{T}_{n-1}^*) = \det \mathbf{B}_{n-1}(z)$. This constitutes the first part of the proof.

We now turn to P(i). We first consider $\widetilde{\mathbf{B}}_n(z)_{1k}$, $\forall k \neq 1$, for $z < 1$. By symmetry of the system, we can in fact focus the proof on $\widetilde{\mathbf{B}}_n(z)_{12}$. Ignoring $\eta_n \mathbf{D}_n$ which is of lower order, as $n \rightarrow \infty$:

$$\widetilde{\mathbf{B}}_n(z)_{12} \sim - \det \begin{pmatrix} -t_1^{*(n)} z & -t_1^{*(n)} z & \cdots & -t_{n-2}^{*(n)} z \\ -t_2^{*(n)} z & 1 - t_0^{*(n)} z & \ddots & \vdots \\ \vdots & \ddots & \ddots & -t_1^{*(n)} z \\ -t_{n-1}^{*(n)} z & -t_{n-3}^{*(n)} z & \cdots & 1 - t_0^{*(n)} z \end{pmatrix}.$$

The key feature that is shared by all the $\widetilde{\mathbf{B}}_n(z)_{1k}$, for $k \neq 1$, is that one of their columns (here the first) contains no element from the diagonal of $\mathbf{B}_n(L)$ (where a 1 appears). Hence

$$\widetilde{\mathbf{B}}_n(z)_{12} \sim - \left(\max_{0 < k < n} |t_k^{*(n)}| z \right) \det \begin{pmatrix} \left[\begin{array}{cccc} \frac{-t_1^{*(n)}}{\max_{0 < k < n} |t_k^{*(n)}|} & -t_2^{*(n)} z & \cdots & -t_{n-2}^{*(n)} z \\ \frac{-t_1^{*(n)}}{\max_{0 < k < n} |t_k^{*(n)}|} & 1 - t_0^{*(n)} z & \ddots & \vdots \\ \vdots & \vdots & \ddots & -t_1^{*(n)} z \\ \frac{-t_{n-1}^{*(n)}}{\max_{0 < k < n} |t_k^{*(n)}|} & -t_{n-3}^{*(n)} z & \cdots & 1 - t_0^{*(n)} z \end{array} \right] \end{pmatrix}.$$

Without loss of generality, we assume for instance that $\max_{0 < k < n} |t_k^{*(n)}| = |t_1^{*(n)}|$. Lemma L(i) shows that for $k \neq 0$, $t_k^{*(n)} = O(n^{-1})$, hence,

$$\widetilde{\mathbf{B}}_n(z)_{12} = O\left(\frac{z}{n}\right) \det \left(\begin{bmatrix} -1 & \mathbf{O}(n^{-1}) \\ \mathbf{O}(1) & \mathbf{B}_{n-2}(z) \end{bmatrix} \right).$$

Using the formula for the determinant of partitioned matrices, the determinant on the right-hand side of the latter expression satisfies, as $n \rightarrow \infty$, $\det(\mathbf{B}_{n-2}(z) + \mathbf{O}(n^{-1})) = O(\det(\mathbf{B}_{n-2}(z)))$, see Abadir and

Magnus (2005, result 12.30). Therefore

$$\widetilde{\mathbf{B}}_n(z)_{12} = O(n^{-1} \det(\mathbf{B}_{n-2}(z))). \quad (\text{S-4})$$

For any polynomial P , let $\deg P$ denote its degree. Now, we introduce the Hadamard polynomial product, which is defined for $P(z) = \sum_{k \geq 0} p_k z^k$ and $Q(z) = \sum_{k \geq 0} q_k z^k$ as $P \circ Q(z) = \sum_{k=0}^{\min(\deg P, \deg Q)} p_k q_k z^k$ and $P(z)^{\circ 2} = P(z) \circ P(z)$. Below, $[P(z)]_{z=1}$ refers to $P(z)$ evaluated at $z = 1$. Hence,

$$\text{Var} \left[\sum_{k \neq j} \frac{\widetilde{\mathbf{B}}_n(z)_{jk}}{\det(\mathbf{B}_n(z))} \epsilon_{kt} \right] = \sum_{k \neq j} \left| \frac{\widetilde{\mathbf{B}}_n(z)_{jk}}{\det(\mathbf{B}_n(z))} \right|_{z=1}^{\circ 2} \sigma_{\epsilon_k}^2, \quad (\text{S-5})$$

where Expression (S-4) implies

$$\sum_{k=2}^n \left| \frac{\widetilde{\mathbf{B}}_n(z)_{1k}}{\det(\mathbf{B}_n(z))} \right|^{\circ 2} = \sum_{k=2}^n O \left([n^{-1}]^2 \left| \frac{\det(\mathbf{B}_{n-2}(z))}{\det(\mathbf{B}_n(z))} \right|^{\circ 2} \right) = O \left(n^{-1} \left| \frac{\det(\mathbf{B}_{n-2}(z))}{\det(\mathbf{B}_n(z))} \right|^{\circ 2} \right). \quad (\text{S-6})$$

The circulant matrix $\mathbf{I}_n - z\mathbf{C}_n$ associated to $\mathbf{B}_n(z)$ has symbol $1 - g(\delta_n, \cdot)z$ since \mathbf{D}_n is antisymmetric. Hence, as $n \rightarrow \infty$, under Assumption T (using the same argument as used in proving Theorem ??),

$$\frac{\det(\mathbf{B}_{n-1}(z))}{\det(\mathbf{B}_n(z))} \sim \frac{\det(\mathbf{I}_{n-1} - z\mathbf{C}_{n-1})}{\det(\mathbf{I}_n - z\mathbf{C}_n)}. \quad (\text{S-7})$$

The limit $(1-z)^{-1/2}$ is finite for $z < 1$ so $\left| \frac{\det(\mathbf{B}_{n-2}(z))}{\det(\mathbf{B}_n(z))} \right|^{\circ 2} = \left| \frac{\det(\mathbf{B}_{n-2}(z)) \det(\mathbf{B}_{n-1}(z))}{\det(\mathbf{B}_{n-1}(z)) \det(\mathbf{B}_n(z))} \right|^{\circ 2}$, i.e. as $n \rightarrow \infty$ $\left| \frac{\det(\mathbf{B}_{n-2}(z))}{\det(\mathbf{B}_n(z))} \right|^{\circ 2} \rightarrow \left| (1-z)^{-1} \right|^{\circ 2}$, where $\left| (1-z)^{-1} \right|^{\circ 2} = \sum_{k=0}^{\infty} |z|^k = (1-|z|)^{-1}$. Hence the expression on the right-hand side in Expression (S-6) is $O(n^{-1})$ when $z \neq 1$.

Now for $z \rightarrow 1$, the truncated polynomial $\left[(1-z)^{-1} \right]^+$ evaluated at $z = 1$ takes the value t so $\sum_{k=2}^n \left| \frac{\widetilde{\mathbf{B}}_n(z)_{1k}}{\det(\mathbf{B}_n(z))} \right|_{z=1}^{\circ 2} = O(n^{-1})$. Together with Expression (S-5), this shows that Assumption P(i) holds.

We now conclude by proving the validity of Assumption P(iv). The assumption follows from Assumption T for \mathbf{T}_n . By construction, \mathbf{T}_n^* is real valued and bounded. By transitivity of asymptotic equivalence (see Gray, 2006, Theorem 2.1),

$$\frac{\det(\mathbf{I}_{n-1} - \mathbf{T}_{n-1}^* z)}{\det(\mathbf{I}_n - \mathbf{T}_n^* z)} \sim \frac{\det(\mathbf{I}_{n-1} - \mathbf{C}_{n-1} z)}{\det(\mathbf{I}_n - \mathbf{C}_n z)} \sim \frac{\det(\mathbf{I}_{n-1} - \mathbf{T}_{n-1} z)}{\det(\mathbf{I}_n - \mathbf{T}_n z)}.$$

Now the circulant matrix associated with $\eta_n \mathbf{D}_n$ has negligible asymptotic entries so $\mathbf{A}_n \sim \mathbf{C}_n$ and the result follows.

4 Proofs relative to Example 2

In this example, many of the results follow from Lemma L and the proofs of Example 1. Assumptions T(i) and T(ii), and T(iii.a) hold as shown in Subsection 3.1. Now $\delta_n = \delta$ for all n so $t_{\delta,k}^{(n)} - t_k^{(n)} \equiv 0$ and Assumption T(iii.b) holds as well. We now consider Assumption P(i)-(iv). The proof of P(ii) follows the lines of the proof provided in Example 1, so $\widetilde{\mathbf{B}}_n(z)_{11} \sim \det(\mathbf{B}_{n-1}(z))$ as $n \rightarrow \infty$. As for Assumption P(i), if $\sigma_{n,k}^2 = o(n^{-1})$ then $\sum_{\substack{k=1 \\ k \neq j}}^n \sigma_{n,k}^2 = o(1)$ so $\sum_{\substack{k=1 \\ k \neq j}}^n \left[\frac{\widetilde{\mathbf{B}}_n(z)_{jk}}{\det(\mathbf{B}_n(z))} \right]_{z=1}^{+\circ 2} \sigma_{n,k}^2 = O \left(\sum_{\substack{k=1 \\ k \neq j}}^n \sigma_{n,k}^2 \right) = o(1)$. Finally P(iii) holds by construction and Assumption P(iv) also holds since $\mathbf{A}_n = \mathbf{T}_n^*$.

5 Proof of Expression (5)

Since $\mathbf{T}_n^* = \text{Re}(\mathbf{T}_n)$, we start expressing the coefficients of the latter matrix.

Since $g(\delta, x) = 1_{\{0 \leq x < \pi\delta\}} + 1_{\{\pi(\frac{3}{2}-\delta) < x \leq \frac{3\pi}{2}\}}$ for $\delta \in (0, 1)$ and $x \in [0, 2\pi]$, the coefficients of \mathbf{T}_n satisfy:

$$\begin{aligned} t_{-k}^{(n)} &= \frac{1}{n} \sum_{\ell=0}^{n-1} 1_{\{\ell < \frac{n\delta_n}{2}\}} e^{2i\pi k\ell/n} + \frac{1}{n} \sum_{\ell=0}^{n-1} 1_{\{(\frac{3n}{4} - \frac{n\delta_n}{2}) < \ell \leq \frac{3n}{4}\}} e^{2i\pi k\ell/n} \\ &= \frac{1}{n} \sum_{\ell=0}^{\lceil n\delta_n/2 \rceil - 1} e^{2i\pi k\ell/n} + \frac{1}{n} \sum_{\ell=\lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor + 1}^{\lfloor \frac{3n}{4} \rfloor} e^{2i\pi k\ell/n}. \end{aligned}$$

Hence, for $k = 0$, $t_0^{(n)} = t_0^{*(n)}$, i.e., $t_0^{(n)} = \frac{\lceil n\delta_n/2 \rceil + \lfloor \frac{3n}{4} \rfloor - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor}{n}$ so

$$\begin{aligned} t_0^{(n)} &= \frac{1}{n} \left(\lceil n\delta_n/2 \rceil - n\delta_n/2 + \left\lfloor \frac{3n}{4} \right\rfloor - \frac{n\delta_n}{2} - \left\lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \right\rfloor \right) \\ &= \frac{1}{n} \left(n\delta_n + \lceil n/4 + o(n^{-1}) \rceil - n/4 + \left\lfloor \frac{3n}{4} \right\rfloor - \frac{n}{4} - \left\lfloor \frac{3n}{4} - \frac{n}{4} + o(n^{-1}) \right\rfloor + o(n^{-1}) \right) \\ &= \delta + (\delta_n - \delta) + \frac{1}{4} \left(\frac{\lceil n/4 \rceil - n/4}{n/4} + o(n^{-1}) \right) - \frac{1}{2} \left(\frac{\lfloor \frac{3n}{4} - \frac{n}{4} \rfloor - \lfloor \frac{3n}{4} \rfloor - \frac{n}{4}}{\lfloor \frac{3n}{4} \rfloor - \frac{n}{4}} + o(n^{-1}) \right) \end{aligned}$$

yielding $t_0^{(n)} = \delta + (\delta_n - \delta) + O(n^{-1})$, and therefore when $n^2(\delta - \delta_n) \rightarrow 0$, $t_0^{*(n)} = \delta + O(n^{-1})$.

Now, when $k \neq 0$,

$$\begin{aligned} t_{-k}^{(n)} &= \frac{1}{n} \frac{1 - e^{2i\pi k \lceil n\delta_n/2 \rceil / n} + e^{2i\pi k (\lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor + 1) / n} - e^{2i\pi k (\lfloor \frac{3n}{4} \rfloor + 1) / n}}{1 - e^{2i\pi k / n}} \\ &= \frac{1}{n} \frac{e^{\frac{i\pi k (2 \lceil n\delta_n/2 \rceil)}{2n}} \left(e^{-\frac{i\pi k (2 \lceil n\delta_n/2 \rceil)}{2n}} - e^{\frac{i\pi k (2 \lceil n\delta_n/2 \rceil)}{2n}} \right)}{e^{\frac{i\pi k}{n}} \left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}} \right)} \\ &\quad - \frac{1}{n} \frac{e^{2i\pi k (\lfloor \frac{3n}{4} \rfloor + 1) / n} \left[1 - e^{-2i\pi k (\frac{n\delta_n}{2} + (\lfloor \frac{3n}{4} \rfloor - \frac{n\delta_n}{2} - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor)) / n} \right]}{e^{\frac{i\pi k}{n}} \left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}} \right)} \end{aligned}$$

so $t_{-k}^{(n)} = \frac{1}{n} \frac{1 - e^{2i\pi k \lceil n\delta_n/2 \rceil / n} + e^{2i\pi k (\lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor + 1) / n} - e^{2i\pi k (\lfloor \frac{3n}{4} \rfloor + 1) / n}}{1 - e^{2i\pi k / n}}$, which yields

$$\begin{aligned} t_{-k}^{(n)} &= \frac{1}{n} \frac{e^{\frac{i\pi k (2 \lceil n\delta_n/2 \rceil)}{2n}} \left(e^{-\frac{i\pi k (2 \lceil n\delta_n/2 \rceil)}{2n}} - e^{\frac{i\pi k (2 \lceil n\delta_n/2 \rceil)}{2n}} \right)}{e^{\frac{i\pi k}{n}} \left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}} \right)} \\ &\quad - \frac{1}{n} \frac{e^{2i\pi k (\lfloor \frac{3n}{4} \rfloor + 1) / n} \left[1 - e^{-2i\pi k (\frac{n\delta_n}{2} + (\lfloor \frac{3n}{4} \rfloor - \frac{n\delta_n}{2} - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor)) / n} \right]}{e^{\frac{i\pi k}{n}} \left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}} \right)} \end{aligned}$$

simplifying further,

$$\begin{aligned}
t_{-k}^{(n)} &= \frac{1}{n} \frac{e^{\frac{i\pi k\delta_n}{2} \left(\frac{[n\delta_n/2]}{\delta_n n/2} \right)} \sin \left\{ \frac{\pi k\delta_n}{2} \frac{[n\delta_n/2]}{\delta_n n/2} \right\}}{e^{\frac{i\pi k}{n}} \sin \frac{\pi k}{n}} \\
&+ \frac{1}{n} \frac{e^{\frac{3i\pi k}{2} \left(1 + \frac{[3n/4] - 3n/4 + 1}{3n/4} \right)} e^{-\frac{i\pi k\delta_n}{2} \left(1 + \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2} \right)} \sin \left\{ \frac{\pi k\delta_n}{2} \left(1 + \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2} \right) \right\}}{e^{\frac{i\pi k}{n}} \sin \frac{\pi k}{n}} \\
&= \frac{e^{\frac{i\pi k\delta_n}{2} \left(\frac{[n\delta_n/2]}{\delta_n n/2} \right)} \left(\sin \left\{ \frac{\pi k\delta_n}{2} \frac{[n\delta_n/2]}{\delta_n n/2} \right\} \right)}{n e^{\frac{i\pi k}{n}} \sin \frac{\pi k}{n}} \\
&+ \frac{e^{\frac{i\pi k(3-\delta_n)}{2}} e^{\frac{3i\pi k}{2} \frac{[3n/4] - 3n/4 + 1}{3n/4}} e^{-\frac{i\pi k\delta_n}{2} \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2}} \sin \left\{ \frac{\pi k\delta_n}{2} \left(1 + \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2} \right) \right\}}{n e^{\frac{i\pi k}{n}} \sin \frac{\pi k}{n}}
\end{aligned}$$

so, finally,

$$\begin{aligned}
t_{-k}^{(n)} &= \frac{\sin \frac{\pi k\delta_n}{2}}{n e^{\frac{i\pi k}{n}} \sin \frac{\pi k}{n}} \left[e^{\frac{i\pi k\delta_n}{2} \left(\frac{[n\delta_n/2]}{\delta_n n/2} \right)} \frac{\sin \left\{ \frac{\pi k\delta_n}{2} \frac{[n\delta_n/2]}{\delta_n n/2} \right\}}{\sin \frac{\pi k\delta_n}{2}} + e^{\frac{i\pi k\delta_n}{2} \left(\frac{[n\delta_n/2]}{\delta_n n/2} \right)} \frac{\sin \left\{ \frac{\pi k\delta_n}{2} \frac{[n\delta_n/2]}{\delta_n n/2} \right\}}{\sin \frac{\pi k\delta_n}{2}} \right. \\
&\left. + e^{\frac{i\pi k(3-\delta_n)}{2}} e^{\frac{3i\pi k}{2} \frac{[3n/4] - 3n/4 + 1}{3n/4}} e^{-\frac{i\pi k\delta_n}{2} \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2}} \frac{\sin \left\{ \frac{\pi k\delta_n}{2} \left(1 + \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2} \right) \right\}}{\sin \frac{\pi k\delta_n}{2}} \right].
\end{aligned}$$

Using $e^{i\left(\frac{3\pi k}{2}-x\right)} = (-1)^k e^{i\left(\frac{\pi k}{2}-x\right)}$, the previous expression can be rewritten as follows:

$$\begin{aligned}
t_{-k}^{(n)} &= \frac{\sin \frac{\pi k\delta_n}{2}}{n e^{\frac{i\pi k}{n}} \sin \frac{\pi k}{n}} \left[e^{\frac{i\pi k\delta_n}{2}} + (-1)^k e^{\frac{i\pi k(1-\delta_n)}{2}} \right] + \zeta_k(\delta_n, n) \\
&= \frac{\delta_n}{2} \frac{\sin_c \frac{\pi k\delta_n}{2}}{e^{\frac{i\pi k}{n}} \sin_c \frac{\pi k}{n}} \left[e^{\frac{i\pi k\delta_n}{2}} + (-1)^k e^{\frac{i\pi k(1-\delta_n)}{2}} \right] + \zeta_k(\delta_n, n),
\end{aligned}$$

where

$$\begin{aligned}
\zeta_k(\delta_n, n) &= \left(e^{\frac{i\pi k\delta_n}{2} \left(\frac{[n\delta_n/2] - n\delta_n/2}{n\delta_n/2} \right)} \right) \left(\frac{\sin \left\{ \frac{\pi k\delta_n}{2} \frac{[n\delta_n/2]}{n\delta_n/2} \right\}}{\sin \frac{\pi k\delta_n}{2}} - 1 \right) \\
&- e^{\frac{i\pi k(1-\delta_n)}{2}} \left(e^{\frac{3i\pi k}{2} \frac{[3n/4] - 3n/4 + 1}{3n/4}} e^{-\frac{i\pi k\delta_n}{2} \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2}} \right. \\
&\left. \times \frac{\sin \left\{ \frac{\pi k\delta_n}{2} \left(1 + \frac{[3n/4] - \frac{n\delta_n}{2} - [3n/4 - \frac{n\delta_n}{2}]}{n\delta_n/2} \right) \right\}}{\sin \frac{\pi k\delta_n}{2}} - 1 \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
t_{-k}^{(n)} &= \frac{\delta_n}{2} \frac{\sin_c \frac{\pi k \delta_n}{2}}{e^{\frac{i\pi k}{n}} \sin_c \frac{\pi k}{n}} e^{\frac{i\pi k}{4}} \left[e^{-\frac{i\pi k(1/2-\delta_n)}{2}} + (-1)^k e^{\frac{i\pi k(1/2-\delta_n)}{2}} \right] + \zeta_k(\delta_n, n) \\
&= \delta_n \frac{\sin_c \frac{\pi k \delta_n}{2}}{\sin_c \frac{\pi k}{n}} e^{\frac{i\pi}{2} \left(\left(\frac{1}{2} - \frac{2}{n} \right) k \right)} \left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}} \sin \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} + 1_{\{k \text{ even}\}} \cos \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} \right] + \zeta_k(\delta_n, n) \\
&= \delta_n e^{i\frac{\pi k}{4}} \sin_c \frac{\pi k \delta_n}{2} \left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}} \sin \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} + 1_{\{k \text{ even}\}} \cos \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} \right] \left(\frac{e^{-i\pi \frac{k}{n}}}{\sin_c \frac{\pi k}{n}} \right) + \zeta_k(\delta_n, n) \\
&= \delta_n e^{i\frac{\pi k}{4}} \sin_c \frac{\pi k \delta_n}{2} \left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}} \sin \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} + 1_{\{k \text{ even}\}} \cos \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} \right] + \xi_k(\delta_n, n) + \zeta_k(\delta_n, n),
\end{aligned}$$

where

$$\begin{aligned}
\xi_k(\delta_n, n) &= \delta_n e^{i\frac{\pi k}{4}} \sin_c \frac{\pi k \delta_n}{2} \left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}} \sin \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} + 1_{\{k \text{ even}\}} \cos \frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} \right] \\
&\quad \times \left(\frac{e^{-i\pi \frac{k}{n}}}{\sin_c \frac{\pi k}{n}} - 1 \right).
\end{aligned}$$

It remains to be shown that both $\xi_k(\delta_n, n)$ and $\zeta_k(\delta_n, n)$ are $O(n^{-1})$. We use the fact that, as $x \rightarrow 0$, $\sin x = x + O(x^3)$, $\sin_c x = 1 + O(x^2)$ and, when $\sin a \neq 0$, $\sin(a+x) = \sin a + x \cos a + O(x^2)$ and $\sin_c(a+x) = \sin_c a + O(x)$. Hence

$$\begin{aligned}
\xi_k(\delta_n, n) &= \left(\frac{1}{2} + O\left(\frac{1}{2} - \delta_n\right) \right) e^{i\frac{\pi(k-2)}{4}} \left(\sin_c \left(\frac{\pi k}{4} \right) + O\left(k \left(\frac{1}{2} - \delta_n \right)\right) \right) \\
&\quad \times \left[\frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} + O\left(k \left(\frac{1}{2} - \delta_n \right)^3\right) \right] \left(\frac{1 + O\left(\frac{k}{n}\right)}{1 + O\left(\frac{k}{n}\right)} - 1 \right) \\
&= \frac{1}{2} e^{i\frac{\pi(k-2)}{4}} \sin_c \left(\frac{\pi k}{4} \right) \left(1 + O\left(\frac{1}{2} - \delta_n\right) \right) \times \left[\frac{\pi k \left(\frac{1}{2} - \delta_n \right)}{2} + O\left(k \left(\frac{1}{2} - \delta_n \right)^3\right) \right] \left(O\left(\frac{k}{n}\right) \right) \\
&= O\left(\frac{k^2}{n} \left(\frac{1}{2} - \delta_n \right)\right) = O\left(n \left(\frac{1}{2} - \delta_n \right)\right),
\end{aligned}$$

and therefore when $n^2(1/2 - \delta_n) \rightarrow 0$, $\xi_k(\delta_n, n) = o(n^{-1})$ while

$$\begin{aligned}
\zeta_k(\delta_n, n) &= \left(e^{\frac{i\pi k \delta_n}{2} \left(\frac{\lfloor n\delta_n/2 \rfloor - n\delta_n/2}{n\delta_n/2} \right)} \left(\frac{\sin \left\{ \frac{\pi k \delta_n \lfloor n\delta_n/2 \rfloor}{2 n\delta_n/2} \right\}}{\sin \frac{\pi k \delta_n}{2}} - 1 \right) \right. \\
&\quad \left. - e^{\frac{i\pi k(1-\delta_n)}{2}} \left(e^{\frac{3i\pi k}{2} \frac{\lfloor 3n/4 \rfloor - 3n/4 + 1}{3n/4}} - e^{\frac{i\pi k \delta_n}{2} \frac{\lfloor 3n \rfloor - \frac{n\delta_n}{2} - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor}{n\delta_n/2}} \right) \right. \\
&\quad \left. \times \frac{\sin \left\{ \frac{\pi k \delta_n}{2} \left(1 + \frac{\lfloor 3n \rfloor - \frac{n\delta_n}{2} - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor}{n\delta_n/2} \right) \right\}}{\sin \frac{\pi k \delta_n}{2}} - 1 \right).
\end{aligned}$$

We note the following expressions

$$\begin{aligned}
e^{\frac{i\pi k\delta_n}{2} \left(\frac{\lceil n\delta_n/2 \rceil - n\delta_n/2}{n\delta_n/2} \right)} &= 1 + O(n^{-1}) \\
\sin \left\{ \frac{\pi k\delta_n}{2} \frac{\lceil n\delta_n/2 \rceil}{n\delta_n/2} \right\} &= \sin \frac{\pi k\delta_n}{2} + O(n^{-1}) \\
e^{\frac{3i\pi k}{2} \frac{\lfloor 3n/4 \rfloor - 3n/4 + 1}{3n/4} - \frac{i\pi k\delta_n}{2} \frac{\lfloor \frac{3n}{4} \rfloor - \frac{n\delta_n}{2} - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor}{n\delta_n/2}} &= 1 + O(n^{-1}) \\
\sin \left\{ \frac{\pi k\delta_n}{2} \left(1 + \frac{\lfloor \frac{3n}{4} \rfloor - \frac{n\delta_n}{2} - \lfloor \frac{3n}{4} - \frac{n\delta_n}{2} \rfloor}{n\delta_n/2} \right) \right\} &= \sin \frac{\pi k\delta_n}{2} + O(n^{-1}).
\end{aligned}$$

Hence, $\zeta_k(\delta_n, n) = (1 + O(n^{-1})) \left(\frac{\sin \frac{\pi k\delta_n}{2} + O(n^{-1})}{\sin \frac{\pi k\delta_n}{2}} - 1 \right) - e^{\frac{i\pi k(1-\delta_n)}{2}} \left([1 + O(n^{-1})] \frac{\sin \frac{\pi k\delta_n}{2} + O(n^{-1})}{\sin \frac{\pi k\delta_n}{2}} - 1 \right)$,
i.e., $\zeta_k(\delta_n, n) = O(n^{-1})$. Now, $t_{-k}^{*(n)} = \operatorname{Re} \left(t_{-k}^{(n)} \right)$ so $t_{-k}^{*(n)} = \delta_n \sin_c \frac{\pi k\delta_n}{2} \left[1_{\{k \text{ odd}\}} \sin \frac{\pi k}{4} \sin \frac{\pi k(\frac{1}{2} - \delta_n)}{2} \right.$
 $\left. + 1_{\{k \text{ even}\}} \cos \frac{\pi k}{4} \cos \frac{\pi k(\frac{1}{2} - \delta_n)}{2} \right] + O(n^{-1})$. Notice that $k(\frac{1}{2} - \delta_n) = o(n^{-1}) \forall k < n$, hence for k odd,
 $t_{-k}^{*(n)} = O(n^{-1})$. When k is even, we need to consider the cases where there exists an odd integer m such
that $k = 4m$ or $k = 4m + 2$. First if $k = 4m$ then $\sin \frac{\pi k\delta_n}{2} = \sin 2\pi m\delta_n = O(m(\frac{1}{2} - \delta_n)) = o(n^{-1})$ and
if $k = 4m + 2$, then $\cos \frac{\pi k}{4} = \cos(m\pi + \frac{\pi}{2}) = 0$. Hence for all k such that $0 < |k| < n$,

$$t_{-k}^{*(n)} = O(n^{-1}),$$

which concludes the proof of Expression (5).